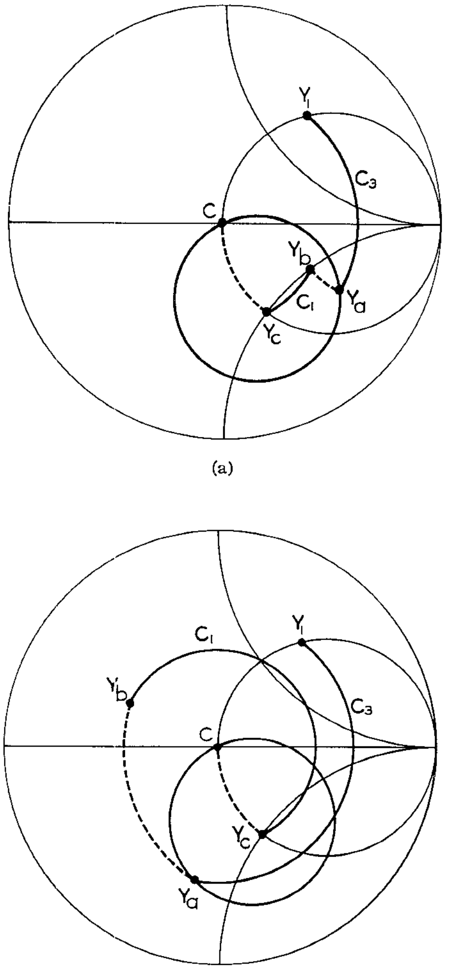

Fig. 6—Diagrams illustrating matching with fixed B.

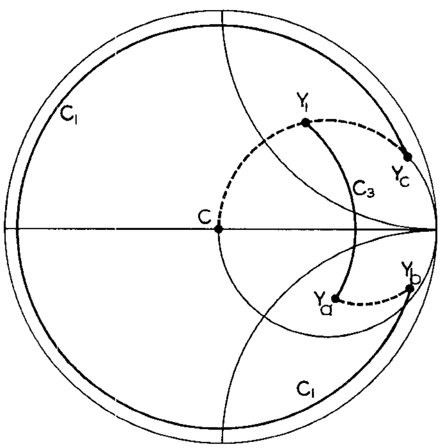

Fig. 7—Diagram illustrating matching with fixed d_1.

$$Y = \frac{2 + B^2}{2} - jB$$

and the radius of C_5 is the same as the radius of C_4. The transformation

$$\Gamma = \frac{Y - 1}{Y + 1}$$

maps the circle C_5 into the circle C_2 in the Γ plane. The center of the circle C_2 is at

$$= \frac{B^2 - i2B}{4 + 2B^2}.$$

The corresponding admittance of the center of C_2 is

$$Y_2 = \frac{1 + \Gamma}{1 - \Gamma}$$

$$= \frac{16 + 12B^2 + 3B^4 - j(16B + 8B^3)}{16 + 12B^2 + B^4}. \quad (1)$$

Let Γ_b and Γ_c denote the current reflection coefficients corresponding to Y_b and Y_c, respectively. The values of Γ_b and Γ_c are given by the equations

$$\Gamma_b = \frac{g + j(b + B) - 1}{g + j(b + B) + 1},$$

$$\Gamma_c = \frac{1 - jB - 1}{1 - jB + 1}.$$

Since Y_b and Y_c lie on the circle C_1,

$$|\Gamma_b|^2 = |\Gamma_c|^2 = \frac{(g - 1)^2 + (b + B)^2}{(g + 1)^2 + (b + B)^2}$$

$$= \frac{B^2}{2^2 + B^2}.$$

This equation can be solved for B to obtain (2). If $g = 1$, it is obvious that $2B = -b$. Thus for a given d_1 and the corresponding $Y_a = g + jb$, there are generally two possible values of B. This is illustrated in Figs. 6(a) and 7 where the value of Y_a is the same but the values of B are different. [For Fig. 6(a), $Y_a = 1.9 - j2$ and $B = 1$. For Fig. 7, $B = -5.4$.]

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Letter from Mr. Reed[6]

Mr. Mathis' theorem is correct but the procedure resulting from this theorem does not give a good result from an engineering standpoint. The result of what he calls two-sided matching will give a match not only at the design center frequency, but also at some other frequency. Thus, the performance curve will *not* be symmetrical about the design center. The procedure suggested in my last note would give a symmetrical curve with maximally-flat response in which the two frequencies of match are the same.

Suppose it is desired to cancel out an inductive iris which has a normalized susceptance of -2. The reflection from this can be cancelled out by the use of another iris whose susceptance is also -2 spaced down the line toward the generator by three-eighths of a wavelength.

Thus, according to his theorem, we can split the matching into two susceptances of -1 on either side of the susceptance of -2 spaced 0.375 λg ($\tan 2\pi s/\lambda g = -1$) of a wavelength from it. But match would also occur if the spacing were 0.3245 λg($\tan 2\pi s/\lambda g = -2$).

For critical coupling $B\sqrt{B^2+4}$ is set equal to -2 and the resulting equation solved for B giving B to -0.91018. This value of B is inserted into the formula for p, thus resulting in this case of a value of p equal to 0.3465λg. See Fig. 8.

[6] Received by the PGMTT, December 19, 1956.

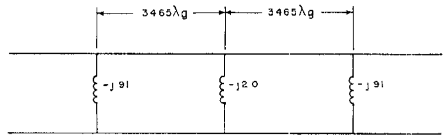

Fig. 8.

Approximate performance curves for one-sided matching, two-sided matching using the Mathis theory, and critically coupled performance as described above are shown below. Some improvements in the critically coupled performance can be obtained by letting the midband be mismatched but be matched on either side of the design frequency. See Fig. 9 below.

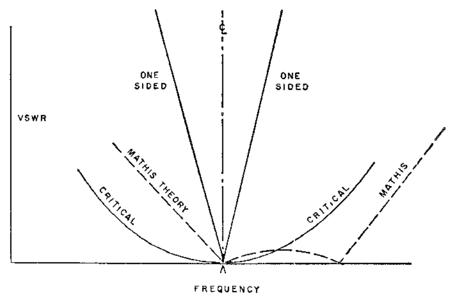

Fig. 9.

JOHN REED

Author's Comment[7]

I agree with the remarks in Mr. Reed's recent note. In my original brief notes, I did not consider the effects of varying the frequency. His two notes are most interesting and valuable. I do not think that I can add anything of value.

H. F. MATHIS

[7] Received by the PGMTT, January 27, 1957.

The Available Power of a Matched Generator from the Measured Load Power in the Presence of Small Dissipation and Mismatch of the Connecting Network*

It is sometimes necessary to determine the available power of a matched generator in terms of the power dissipated in a load when the load is connected to the generator by means of a slightly mismatched 4-pole having small loss. (A piece of waveguide or short length of coaxial line could exemplify such a 4-pole; the discontinuities at flanges or at connectors and supporting beads could give rise to the slight mismatch.)

* Received by the PGMTT, October 1, 1956. The research reported in this document has been made possible through support and sponsorship extended by the Rome Air Dev. Ctr., Contract AF-30(602)-988. It is published for technical information only and does not represent recommendations or conclusions of the sponsoring agency.

Referring to Fig. 1,

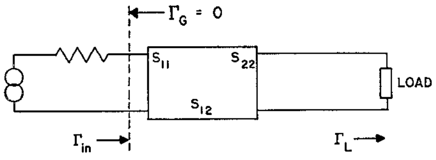

Fig. 1.

let

Γ_L = the reflection coefficient of the load
Γ_{in} = the reflection coefficient of the 4-pole and load
S_{11}, S_{22}, S_{12} = the scattering coefficients of the 4-pole
P_0 = the available power of the generator
P_L = the power dissipated in the load

$$P_L = KP_0. \qquad (1)$$

The constant K may be expressed in terms of all or some of the scattering coefficients and only one of the reflection coefficients—either Γ_L or Γ_{in}. Thus,

$$K = \frac{|S_{12}|^2[1 - |\Gamma_L|^2]}{|1 - S_{22}\Gamma_L|^2} \qquad (2)$$

$$K = \frac{|S_{12}^2 + S_{22}(\Gamma_{in} - S_{11})|^2 - |\Gamma_{in} - S_{11}|^2}{|S_{12}|^2}. \qquad (3)$$

The question arises, which expression should be used to determine K if only the magnitudes (but not the phases) of the various scattering and reflection coefficients are known. If $|\Gamma_L|$ is known (and since the phase of Γ_L is arbitrary) the possible range of K may be determined from the maximum and minimum values of K given by (2), namely,

$$K_{max} = \frac{|S_{12}|^2[1 - |\Gamma_L|^2]}{[1 - |S_{22}\Gamma_L|]^2} \qquad (4)$$

$$K_{min} = \frac{|S_{12}|^2[1 - |\Gamma_L|^2]}{[1 + |S_{22}\Gamma_L|]^2}. \qquad (5)$$

(It is interesting to note that the db difference between K_{max} and K_{min},

$$20 \log \frac{1 + |S_{22}\Gamma_L|}{1 - |S_{22}\Gamma_L|},$$

is independent of S_{12}^2 and increases almost linearly with $|S_{22}\Gamma_L|$ for small values of this product.)

When $|\Gamma_{in}|$ is known (and is the only reflection coefficient accessible to direct measurement because the load is an integral part of a structure which cannot be readily taken apart) (3) must be used to estimate the possible range of K. If the worst possible phase combinations of the coefficients are used in the estimation, an unnecessarily large uncertainty in K will result if the phases of S_{11}, S_{22}, and S_{12} are assumed completely independent. In actuality the phases are restricted by the following relation if the 4-pole is to be physically realizable.[1]

[1] R. LaRosa and H. J. Carlin, "A general theory of wideband matching with dissipative 4-poles," *J. Math. and Phys.*, vol. 33, pp. 331–345; January, 1955.

TABLE I

Results Calculated from (2) for $|\Gamma_L| = 0.15$

	$\|S_{12}\|^2 = 0.98$				$\|S_{12}\|^2 = 0.95$				$\|S_{12}\|^2 = 0.90$			
$\|S_{11}\|$	K_{max}	K_{min}	K_0	Error	K_{max}	K_{min}	K_0	Error	K_{max}	K_{min}	K_0	Error
0.03	0.968	0.950	0 958	0.010	0.938	0.920	0.929	0.009	0.889	0.872	0.880	0.009
0.15	0.973	0.944	0.958	0.015	0.943	0.915	0.929	0.014	0.893	0.868	0.880	0.013
0.10	0.988	0.930	0.958	0.030	0.957	0.901	0.929	0.029	0.907	0.854	0.880	0.027

TABLE II

Results Calculated from (3) for $|\Gamma_{in}| = 0.15$

	$\|S_{12}\|^2 = 0.98$				$\|S_{12}\|^2 = 0.95$				$\|S_{12}\|^2 = 0.90$			
$\|S_{11}\|$	K_{max}	K_{min}	K_0	Error	K_{max}	K_{min}	K_0	Error	K_{max}	K_{min}	K_0	Error
0.03	0.964	0.952	0.957	0.007	0.937	0.916	0.926	0.011	0.892	0.852	0.875	0.023
0.05	0.965	0.954	0.957	0.008	0.942	0.912	0.926	0.016	0.899	0.836	0.875	0.039
0.10	0.970	0.964	0.957	0.013	0.947	0.922	0.926	0.021	0.905	0.847	0.875	0.030

$$(1 - |S_{12}|^2)^2 - |S_{22}|^2 - |S_{11}|^2 + |S_{11}S_{22}|^2 = 2 \text{ Re } S_{12}^{*2}S_{11}S_{22}. \qquad (6)$$

The maximum and minimum values of K should be computed from (3) taking into account the above restriction.

If $|S_{11}|$ is assumed equal to $|S_{22}|$, (3) may be transformed to

$$K = \frac{k_0 + k_1(\Phi, \psi)}{|S_{12}|^2} \qquad (7)$$

where

$\psi = \arg (S_{12}^{*2}S_{11}S_{22})$
$\Phi = \arg (\Gamma_{in}^* S_{11})$
$k_0 = |S_{12}|^4 + |S_{11}|^4 - |S_{11}|^2 - |\Gamma_{in}|^2$
$k_1(\Phi, \psi) = A \cos \Phi + B \cos (\psi - \Phi) - C \cos \psi$
$A = 2(1 - |S_{11}|^2)|S_{11}\Gamma_{in}|$
$B = 2|S_{12}|^2|S_{11}\Gamma_{in}|$
$C = 2|S_{12}^2S_{11}S_{22}|$.

Eq. (6) then becomes

$$(1 - |S_{12}|^2)^2 - 2|S_{11}|^2 + |S_{11}|^4 \geq C \cos \psi. \qquad (8)$$

Although Φ is an unrestricted angle, ψ is constrained by the above inequality to the region $\pi - \alpha \leq \psi \leq \pi + \alpha$, where α is a positive number $\leq \pi$ whose value depends on $|S_{12}|^2$ and $|S_{11}|^2$. The extreme values of $k_1(\Phi, \psi)$ determine the extremes of k. The former can be obtained in the usual way by setting $\partial k_1/\partial\Phi$ and $\partial k_1/\partial\psi$ equal to zero and solving for the corresponding values of Φ and ψ. If a solution lies within the permissible region for Φ and ψ the corresponding maxima and minima of k_1 are evaluated. In addition solutions for the maxima and minima of k_1 on the boundary of the permissible region ($\psi = \pi - \alpha$, $\pi + \alpha$) are obtained by setting $\partial k_1/\partial\Phi$ equal to zero on the boundary. The extreme values thus obtained are compared with the values (if any) which lie within the region and the most extreme values are used in calculating k_{max} and k_{min}.

The values of K_{max} and K_{min} have been tabulated for two types of cases for typical values of the scattering coefficients. In the first type (Table I) $|\Gamma_L|$ is assumed known and (2) is used in estimating the uncertainty in K. In the second type (Table II) $|\Gamma_{in}|$ is assumed known, $|S_{11}|$ is assumed equal to $|S_{22}|$ and (3) is used subject to the restriction of (6). The uncertainty in K may be expressed by the difference between K_{max} or K_{min} and an intermediate value K_0, obtained by setting $|S_{11}| = |S_{22}| = 0$. By (2) and (3) respectively

$$K_0 = \frac{|S_{12}|^4 - |\Gamma_{in}|^2}{|S_{12}|^2} \qquad (9)$$

$$K_0 = |S_{12}|^2[1 - |\Gamma_L|^2]. \qquad (10)$$

The importance of the physical realizability criterion can best be appreciated by considering an example. Let $|S_{11}| = |S_{22}| = 0.1$, $|S_{12}|^2 = 0.95$ and $|\Gamma_{in}| = 0.15$. For the worst phase combinations (3) gives a maximum difference between K and K_0 of about 9 per cent if the phases are unrestricted. This is reduced to 2 per cent if the restriction of (6) is applied.

In most cases K_0 is not much different from a value midway between K_{max} and K_{min} and may therefore be used to represent K with minimum error. The only 4-pole parameter required to determine K_0 is the attenuation, $|S_{12}|^2$, of the 4-pole. For a structure such as a short piece of waveguide $|S_{12}|^2$ is usually a smooth and slowly varying function of frequency which can be determined once and for all.

In measuring P_L the coefficient $|\Gamma_{in}|$ can generally be determined at the same time, whereas $|\Gamma_L|$ must be derived from previous measurements, generally by interpolation between data points. Eq. (9) will therefore be more useful than (10). Moreover, for small attenuations ($|S_{12}|^2 \geq 0.95$), (9) will yield a smaller uncertainty in K than (10). For larger attenuations (9) rapidly loses its usefulness and (10) yields smaller uncertainties. However, if interpolated values of $|\Gamma_L|$ are used and $|\Gamma_L|$ is an erratic function of frequency these uncertainties may be appreciably increased.

Acknowledgment

The aid and suggestions of Dr. H. J. Carlin of the Microwave Research Institute are gratefully acknowledged.

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